Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Normal Impingement of Jets

V. Parameswaran* and S. A. Alpay†
University of Waterloo, Ontario, Canada

Nomenclature

AB = nozzle width
$\mathbf{c}$ = general vector
C_E, C_W, C_S, C_N = Laplacian coefficients
h_E, h_W, h_S, h_N = grid sizes
k = level of iteration
p = pressure
u = velocity at any point along the flow direction
v = velocity at any point normal to the flow direction
u_+ = nondimensional velocity
u_{max} = maximum velocity at any cross section
x, y = coordinate axes
$\dot{\mathbf{x}}$ = material velocity
$\ddot{\mathbf{x}}$ = material acceleration
y_+ = nondimensional y distance
xy = perpendicular surface
σ = goertler constant
Ω = vorticity
ν = force potential
ρ = density
ψ = stream function
ψ^* = nondimensional stream function
ϵ = small number
∇^2 = Laplacian

Introduction

THE impingement of jets on a plane surface perpendicular to the jet axis is one of the important problems in the study of jets. The flow in normal impinging jets has attracted the attention of a great number of investigators, who carried out extensive theoretical and experimental studies. A comprehensive survey of the subject can be found in Murdin.[1] The concerted effort in the analysis of normal impinging jets is motivated by its diverse and intensive applications in the various industries encompassing satellites, airplanes, ground cushion vehicles, fluidic elements and many more. The interest in this study is not only restricted to fluid dynamics, it extends itsel also to heat transfer problems, for example, it has been found out that substantially increased heat transfer takes place in the vicinity of the stagnation point of jets.[5] Among many others, some previous recent studies are worthwhile to quote due to their connection to this paper.

Tani and Komatsu[2] have investigated the impingement of a jet experimentally. Bradbury[3] conducted experiments on a jet impinging on ground at various distances, Foss and Kleis[4] studied the oblique impingement of jets. Wolfshtein[5] obtained solution to the problem of impinging jets by a numerical solution for the full Navier Stokes equations. Schauer and Eustis[6] have measured the surface static pressure and maximum velocity in the impinging region. The objective of the present study is to obtain a simple engineering solution enabling one to compute quickly the value of relevent variables without having recourse to such complicated analysis as the full numerical solution of Navier Stokes equations.

Analysis

Figure 1 shows the flow situations analyzed. A jet of fluid emanating from a nozzle AB impinges on to a flat plate xy posed perpendicular to it. Since the flow turns through a right angle after impingement there is no predominant flow direction. There the familiar and well developed method of boundary layers is inapplicable. In the past the analysis of such a flow was obtained either by the full solution of Navier Stokes equation or by using the well known methods of complex variables. The application of the first method for numerous problems requires a large amount of computer time and involves truncation errors and stability problems.[7] The second method assumes irrotational flow which is rarely encountered since the upstream history of the jet which is accompanied by the development of viscous forces, both laminar and turbulent, produces a finite amount of rotation in the flow.

In the present analysis the flow considered is divided into two distinct regimes whose solutions can be found with much more ease. As shown in Fig. 1, the flow region is divided into a free jet region and an impinging region. In the free jet region there is a predominant flow direction and the boundary layer equations hold. Thus at any cross section normal to the predominant flow direction one obtains the velocity distribution in the usual manner,[10] by the familiar Goertler profile given by

$$u/u_{max} = \text{sech}^2(\sigma y/x) \tag{1}$$

when u is the velocity at the cross section u_{max} the maximum velocity, σ the spread parameter x and y are the streamwise and normal coordinates.

In the impingement region there is a change of flow direction and thus the boundary layer approximation is invalid. However, since the change of direction is rapid the convection forces predominate over the viscous forces and it is possible to use the inviscid flow approximations. This is nearly valid everywhere except in a very thin region near the wall where no slip conditions have to be satisfied. Thus the method of solution to be adopted may be analogous to the formation of the boundary layer whose free stream conditions are the solutions of the rotational inviscid analysis. Van Dyke,[8] through the concepts of perturbation analysis has outlined the use of inviscid approximations in cases where shear flows turns around surfaces of very large curvatures. Also as outlined earlier, the potential flow solutions alone are inapplicable since there is a finite amount of vorticity that is induced in the flow through Eq. (1). The rotation however, can be incorporated by considering it as "frozen" along the path of streamlines. A proof of this statement and a precise definition of the "frozen vorticity" will be developed in the following derivation.

Received November 8, 1973. The writers desire to express their thanks to National Research Council of Canada, who supported this work under Grant A1644.

Index category: Jets, Wakes, and Viscid-Invisced Flow Interactions.

*Postdoctoral Fellow.

†Professor of Mechanical Engineering.

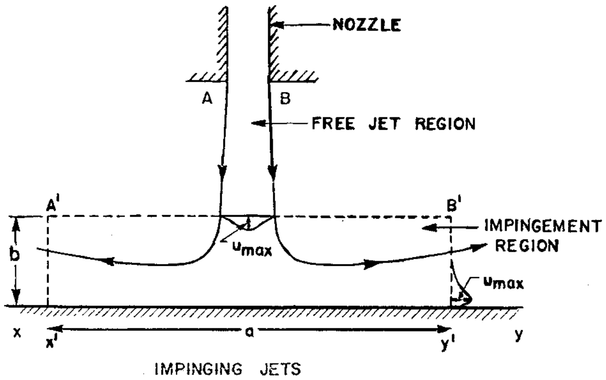


Fig. 1 Impinging jets.

According to Truesdell,[9] the equation for a permanent vector line c in a fluid flow is given by the vector equation

$$c \times [(\partial c/\partial t) + \text{curl}\,(c \times \dot{x}) + \dot{x}\ \text{div}\, c\,] = 0 \qquad (2)$$

If streamlines are to coincide with the vectorline then Eq. (2) is satisfied by replacing **c** by $\dot{\mathbf{x}}$ leading to the condition

$$\dot{x} \times (\partial \dot{x}/\partial t) = 0 \qquad (3)$$

In the present case since only steady flows are considered $\partial\, \dot{\mathbf{x}}/\partial t = 0$ and Eq. (3) is satisfied. If further the vorticity lines are to coincide with vector line **c** then Eq. (2) is satisfied where **c** is replaced by Ω

$$\Omega \times [\partial\Omega/\partial t + \text{curl}(\Omega \times \dot{x})] = 0 \qquad (4)$$

Lagrange's acceleration formula[9] gives for acceleration $\ddot{\mathbf{x}}$

$$\ddot{x} = \partial\dot{x}/\partial t + \Omega \times \dot{x} + \text{grad}\ (1/2\ \dot{x}\ .\ x) \qquad (5)$$

Taking the curl of both sides

$$\text{curl}\ \ddot{x} = (\partial\Omega/\partial t) + \text{curl}\ (\Omega \times \ddot{x}) \qquad (6)$$

leading to:

$$\Omega \times \text{curl}\,\ddot{x} = 0 \qquad (7)$$

If the fluid is subjected to only conservative external body forces acceleration $\ddot{\mathbf{x}}$ is given by

$$\ddot{x} = -\,\text{grad}\ \nu - (1/\rho)\ \text{grad}\ p$$

where ν is the potential of force and p the pressure, this gives finally

$$\text{curl}\ \ddot{x} = 0$$

Thus in the present case Eq. (7) is satisfied and vector line coincides with streamlines and vorticity lines. The physical interpretation of this result may be summarized in stating that vorticity is conserved along streamlines and simply convected if it were a rigid body without any diffusion. This type of vorticity was termed as "Frozen Vorticity" by Robertson.[11]

The expression for vorticity in two-dimensional flows is

$$\Omega = (\partial v/\partial x) - (\partial u/\partial y) \qquad (8)$$

Introducing a stream function ψ defined by $\partial\psi/\partial y = u$, $\partial\psi/\partial x = -v$, we have

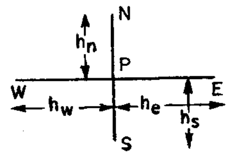
$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \Omega = 0 \qquad (9)$$

Since vorticity is constant along a streamline a functional relationship between the two would be satisfied at every point in the flow. This is usually obtained by using the value of the vorticity produced during the previous history of flow obtained by Eq. (1)

$$u_+ = \text{sech}^2 y_+$$

where $u_+ = u/u_{\max}\ y_+ = (\sigma y/x)$. At the inlet boundary $\Omega = (\partial v/\partial x) - (\partial u/\partial y)$. since $(\partial v/\partial x) << (\partial u/\partial y)$ at that section owing to the boundary layer approximation

Fig. 2 Distribution of grid.



$$\Omega = -\ (\partial u/\partial y)$$

Thus

$$\Omega = -\ (\partial u/\partial y) = +\ u_{\max}(\sigma/x)2\ \text{sech}^2 y_+ \tan h y_+ \qquad (10)$$

$$\psi = \int_0^y u dy = u_{\max}\ \frac{x}{\sigma}\ \tan h(\sigma y/x) \qquad (11)$$

$$= \Omega = 2(u_{\max}\,\sigma/x)\tan y_+(1 - \tan y_+^2)$$

$$= 2(u_{\max}\sigma/x)(\psi\sigma/u_{\max})(1 - (\psi\sigma/y) \qquad (12)$$

Defining $\psi_+ = (-\psi\sigma/xu_{\max})$ Eq. (9) yields

$$(\partial^2\psi_+/\partial x^2) + \partial^2\psi_+/\partial y^2 + 2(\frac{\sigma}{x})^2\psi_+\{1 - \psi_+^2\} = 0 \qquad (13)$$

If $\lambda = 2(\sigma/x)^2$ we reach at the end

$$\Delta^2\psi_+ + \lambda\{1 - \psi_+^2\}\psi_+ = 0 \qquad (14)$$

Solution

The Eq. (14) is nonlinear and could be solved by the method of Green's function as outlined in Ref. 7. However, in the present paper a finite difference solution will be attempted. Considering the field of interest as to be divided by a network of grids produced by horizontal and vertical lines (Fig. 2) the finite difference approximation yields

$$C_E\psi_E + C_W\psi_W + C_N\psi_N + C_S\psi_S - (\textstyle\sum C)\psi_p + \lambda(1 - \psi_p^2)\psi_p = 0 \qquad (15)$$

where c's are the Laplacian coefficients given by

$$C_E = (2/(h_W + h_E)h_E) \qquad C_W = (2/(h_E + h_W)h_W)$$

$$C_S = (2/(h_S + h_N)h_S) \qquad C_N = (2/(h_S + h_N)h_N)$$

$$\textstyle\sum C = C_E + C_W + C_S + C_N$$

The above equation is applied to each and every point in the grid which yields a matrix equation that was solved by the method of successive line over-relaxation technique yielding to

$$-\ C_E\psi_E - C_W\psi_W + \psi_p(\textstyle\sum C + \lambda(\psi p^{2(k-1)} - 1)$$

$$= C_N\psi_N^{(k-1)} + C_S\psi_S^{(k-1)} \qquad (16)$$

where $(k - 1)$ refers to the $(k - 1)$th iteration level. Treating the right hand side as known at the $(k - 1)$th level, the equation could be solved by solving the tridiagonal matrix Eq. (16). Iteration will be carried out till

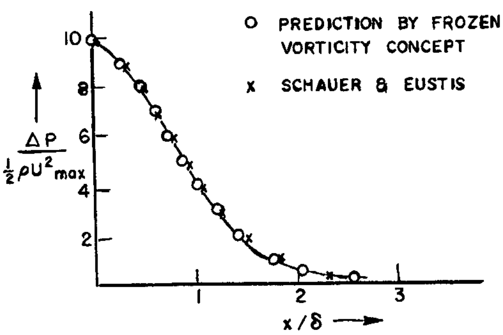


Fig. 3 Prediction of static pressure distribution by frozen vorticity concept.

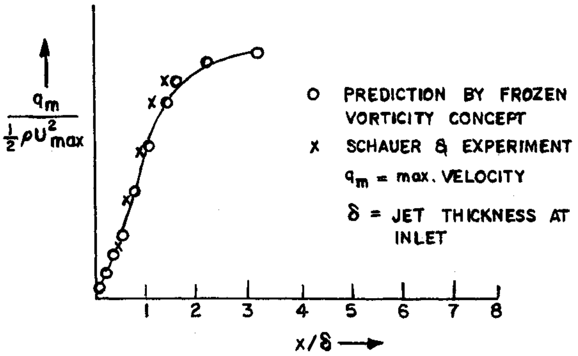


Fig. 4 Prediction of static pressure distribution by frozen vorticity concept.

$$\left| \frac{\psi_p^k - \psi_p^{k-1}}{\psi_{max}} \right| < \epsilon$$

where ϵ is a small number. ϵ in the present study was chosen to be 10^{-4}.

Boundary Conditions

The inlet boundary conditions on $A'B'$ are fixed by upstream velocity profile given by Eq. (1). The wall boundary conditions are fixed by $\psi = 0$. At the outlet $A'x'$ and $B'y'$ it was assumed that boundary layer conditions were met again gives a profile given by Eq. (1).

Results

Figures 3 and 4 illustrate the plot of the velocity maxima predicted by the above method compared with the measurements of Schauer and Eustis.[6] The nondimensional static pressure on the bottom wall is also calculated by using Bernoulli's theorem.

$$p + 1/2\rho u^2 = \text{const}$$

on a streamline. The agreement with the experiments of Refs. 3 and 6 is good for the measured static pressure distribution. It is also seen from the above graphs that a nondimensional profile could be obtained for various nozzle distances. The computing time involved in the above calculations were of the order of 5 secs. The boundary layer close to the wall could be described by a Boundary layer type approximation with a velocity at the outer edge given by the velocity distribution on the bottom plate.

Conclusions

Considering the normal impinging jet flow as consisting of two different regimes an expeditive method of calculation has been developed. The present method takes into account the viscosity effects producing a rotation in the flow, and also it requires an incomparably shorter computer time than needed for the solution of the full Navier-Stokes equation. The accuracy of the predictions by using this method is as good as the predictions assumed by more time consuming methods.

References

[1]Murdin, P. M., "Flow Surveys of a Jet Impinging Normally on to a Plane Surface," RAE TR 68024, 1968, Royal Aircraft Establishment, Farnborough, England

[2]Tani, I. and Komatsu, Y., "Impingement of a Round Jet on a Flat Plate," *Eleventh International Congress of Applied Mechanics,* Munich, 1964, Springer-Verlag, Berlin, 1966, pp. 672–676.

[3]Bradbury, L. J. S., "The Impact of an Axysymmetric Jet on to a Normal Ground," *Aeronautical Quarterly,* May 1972, pp. 141–147.

[4]Foss, J. F. and Kleis, J. S., "The Oblique Impingement of an Axysymmetric Jet," *Proceedings of the Fourth Canadian Congress of Applied Mechanics,* May 28–June 9, 1973, pp. 717–718.

[5]Wolfshten, M., "Convection Process in Turbulent Impinging Jets," Ph.D. thesis, Imperial College, 1967, Dept. of Mechanical Engineering, London, England.

[6]Schauer, J. J. and Eustis, R. H., "The Flow Development and Heatr Transfer Characteristics of Plane Turbulent Impinging Jet," TR 3, 1963, Stanford Univ., Mechanical Engineering Dept., Stanford, Calif.

[7]Parameswaran, V., "Study of Reattaching Wall Jets," Ph.D. thesis, 1973, Mechanical Engineering Dept., University of Waterloo, Ontario, Canada.

[8]Van Dyke, M., "Higher-Order Boundary-Layer Theory," *Annual Review of Fluid Mechanics,* Vol. 1, 1969, pp. 265–292.

[9]Truesdell, C. A., *Kinematics of Vorticity,* Indiana University Press Bloomington, Ill. 1954, pp. 86–87.

[10]Abramovich, G. N., *Theory of Turbulent Jets,* MIT Press, Cambridge, Mass., 1957, pp. 109.

[11]Robertson, J. M. and Huang, W. H. "Rotational flow analysis of plane right angle bends." *Journal of Applied Mathematics and Physics,* Vol. 18, Fasc 5, 1967, pp. 621–642.

On Calculation of Induced Drag and Conditions Downstream of a Lifting Wing

W. R. Sears*

Cornell University, Ithaca, N. Y.

THIS Note concerns the calculation of the induced drag of a wing in terms of conditions far behind it—in the "Trefftz Plane." As every student of wing theory is taught, the lift can easily be calculated by momentum principles: it is equal to the rate of increase of downwash momentum (more precisely, *impulse*) and therefore to the downward impulse in a unit slab of two-dimensional flow downstream. But when the drag is sought it is customary to resort to an energy argument, and it is easily ascertained that the drag of any wing in steady flight is equal to the kinetic energy in the same unit slab downstream.[1,2] It is instructive, however, to carry out the drag calculation by momentum principles; the calculation has certain subtleties and casts light on some interesting facts concerning the flow downstream of a lifting wing-system—a topic of considerable current interest.

The subtleties mentioned arise from the fact that, for a lightly loaded wing (small-perturbation flow), the drag is a second-order quantity. It is necessary to account for the first-order deflection of the wake; hence, the configuration under consideration is that sketched in Fig. 1, which shows the wing and trailing-vortex wake in a wing-fixed frame of reference (steady flow).

At this point we fix our attention specifically on the wing of elliptic lift distribution, for which the downwash at the vortex sheet far downstream is uniform across the span and equal to $(2C_\beta/\pi R)V$ or $4_\beta/\pi\rho Vn^2$, where C_β denotes the lift coefficient, R the aspect ratio b^2/S, L the lift, ρ the fluid density, b the wing span, and S the wing area. To emphasize a peculiarity of this situation, however, we shall replace the factor 2 here by a symbol k,

---

Received January 7, 1974.

Index categories: Aircraft Aerodynamics (Including Component Aerodynamics).

*Professor.